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The Phase Diagram of a Spin Glass Model

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A spin glass problem on a Cayley tree with ferromagnetic interactions is solved rigorously. Using a level-I large deviation argument together with the martingale approach used by Buffet, Patrick and Pulé,[1] explicit expressions for the free energy are derived in different regions of the phase diagram. It is found that there are four phases: a paramagnetic phase, a spin-glass phase, a ferromagnetic phase and a mixed phase. The nature of the phase diagram depends on the power with which the ferromagnetic term occurs in the Hamiltonian.

KEY WORDS: Spin glasses; directed polymers; martingales; large deviations; information theory.

1. THE DIRECTED POLYMER PROBLEM AND THE GENERALIZED RANDOM ENERGY MODEL

The problem of a spin glass on a Cayley tree (or equivalently, a directed polymer) is one of a handful of models in disordered systems that can be solved exactly. It is a simplification of the more realistic case where one considers a regular lattice in place of the Cayley tree. The problem has been treated for instance using the replica method,[5] using the properties of Generalized Random Energy Model[4, 2] by reducing the problem to a reaction-diffusion system[8] and by a martingale approach.[1] The latter

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approach is particularly elegant and achieves a completely rigorous and transparent solution to the problem. Here we use a combination of the martingale approach of ref. 1 and a level-1 large deviations argument[10, 11, 15] to solve a spin glass model on a Cayley tree with an additional mean-field ferromagnetic interaction term in the Hamiltonian. We consider a one-parameter family of such models distinguished by the power $p \geqslant 2$ to which this term is raised and show that the phase diagram in the case $p > 2$ is qualitatively different from that in the case $p=2$. The two phase diagrams are depicted in Fig. 2(a and b). We derive completely rigorously a variational expression for the free energy of our model and then analyze carefully the various regions of the phase diagram. We note that the free energy of the Generalized Random Energy Model in a magnetic field has been computed by Derrida and Gardner[6] using the replica method, which is of course, not rigorous. Our result (Section 2) has a direct analogy with theirs (Section 4 in ref. 6) in this case.

The spin glass on a Cayley tree is in fact similar to Derrida's Generalized Random Energy Model, which has been used in various applications, notably information theory[13, 14] and neural networks.[6, 7] It follows that our results may have implications for applications in these areas. In particular, we have outlined the implications of our results for the optimal decoding problem as proposed by Sourlas[13, 14] in a separate paper.[9] Indeed, it turns out that the phase diagram is identical to that of the Random Energy Model with the same ferromagnetic interaction term as above. We claim that this model is relevant for Sourlas' decoding theory in the case of large p. Indeed, in refs. 3 and 4 Derrida already showed that the random energy model is the limit of the Sherrington–Kirkpatrick model with p-spin interaction. As explained in ref. 9, Sourlas' coding scheme amounts to adding a p-spin Ising term, the ground state of which corresponds to the original message. Random noise in the transmission line then leads to a Random Energy Model with p-spin interaction in the limit $p \to \infty$.

Let us now define the model and the terminology we adopt in this paper: Consider a **Cayley tree** (cf. Fig. 1) with *co-ordination number* 3—i.e., each node of the tree is connected to another two at the next level. Label the bonds of the tree by (j, k) where $j, k \in \mathbb{N}$ and j corresponds to the generation and $k \in \{1, ..., 2^j\}$ labels the bonds from left to right within the jth generation. To each bond of the tree attach i.i.d random variables $V_{j,k}$ with distribution depending on a parameter γ.

A **path** of length n starting at the top of the tree is defined as a finite sequence

$$\{(j, k_j); 1 \leqslant j \leqslant n\} \tag{1.1}$$

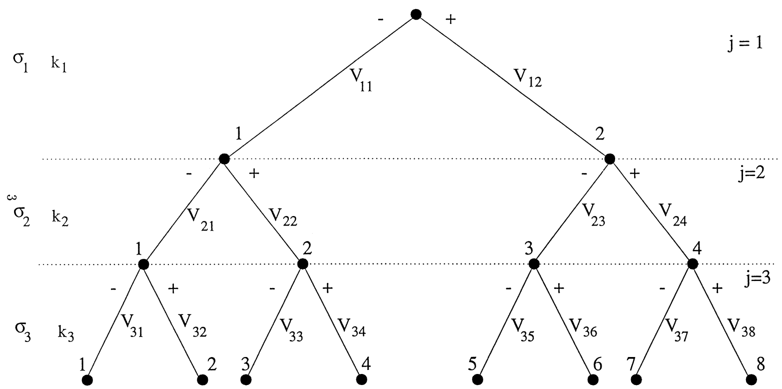

Fig. 1. $k_{j+1}=2k_j-(1-\sigma_{j+1})/2$. $k_i=1, 2, \ldots$ and σ_i take ± 1.

satisfying the relation

$$k_{j+1}=2k_j-\tfrac{1}{2}(1-\sigma_{j+1}) \tag{1.2}$$

where $\sigma_j \in \{-1, 1\}$ correspond to taking the left or right branch out of generation j. (see Fig. 1). Denote by $(\sigma)^j$ the sequence of Ising spins $\{\sigma_k\}_{k=1}^{j}$. Then the path is completely determined by $(\sigma)^n$. Define the Hamiltonian by

$$-\mathscr{H}=\sum_{j=1}^{n} V_{j,(\sigma)^j}+\frac{\lambda}{n^{p-1}}\left|\sum_{j=1}^{n}\sigma_j\right|^p \tag{1.3}$$

where $p \geqslant 2$ is an arbitrary parameter and $\lambda > 0$ is a coupling constant. The **partition function** is defined by

$$\mathscr{Z}_n=\sum_{\{\sigma_j\}_{j=1}^{n}} \mathrm{e}^{-\beta\mathscr{H}} \tag{1.4}$$

The **(specific) free energy** of the model is defined by

$$-\beta f(\beta, \lambda, \gamma)=\lim_{n\to\infty}\frac{1}{n}\log\mathscr{Z}_n(\beta) \tag{1.5}$$

In Section 4 we show that this limit exists almost surely with respect to the random variables V and we derive expressions for it in the cases $p=2$ and $p > 2$ respectively. The phase diagram consists of four different phases: the paramagnetic phase (P), a spin-glass phase (SG), a ferromagnetic phase (F) and a mixed phase (M). The phase diagram in the case $p=2$ is depicted in

Fig. 2(a); and that in the case $p > 2$ in Fig. 2(b). With reference to Fig. 2(a), in paramagnetic regime (P), where $\lambda < \mathscr{C}_1(\beta)$ and $\beta < \beta_0$, and also in region (SG) where $\beta > \beta_0$, $\lambda < \lambda_0$, the magnetization $m=0$. In the latter phase the free-energy remains constant and in the absence of long-range order this is a spin glass or frozen phase. The region (F) where $\mathscr{C}_1(\beta) < \lambda < \mathscr{C}_2(\beta)$ and $m \neq 0$ is the ferromagnetic phase. (M) is a mixed phase where $\lambda > \mathscr{C}_2(\beta)$ or $\lambda > \lambda_0$ and the magnetization $m \neq 0$ depends only on λ.

In Fig. 2(b), the effect of the higher order ferromagnetic term in (1.3) is visible from the curve $\mathscr{C}$ in contrast to the case $p=2$. Indeed, it will be shown that as p increases the points A and D in Fig. 2(b) drift apart. We will also show that for $p > 2$ the magnetization is discontinuous across the

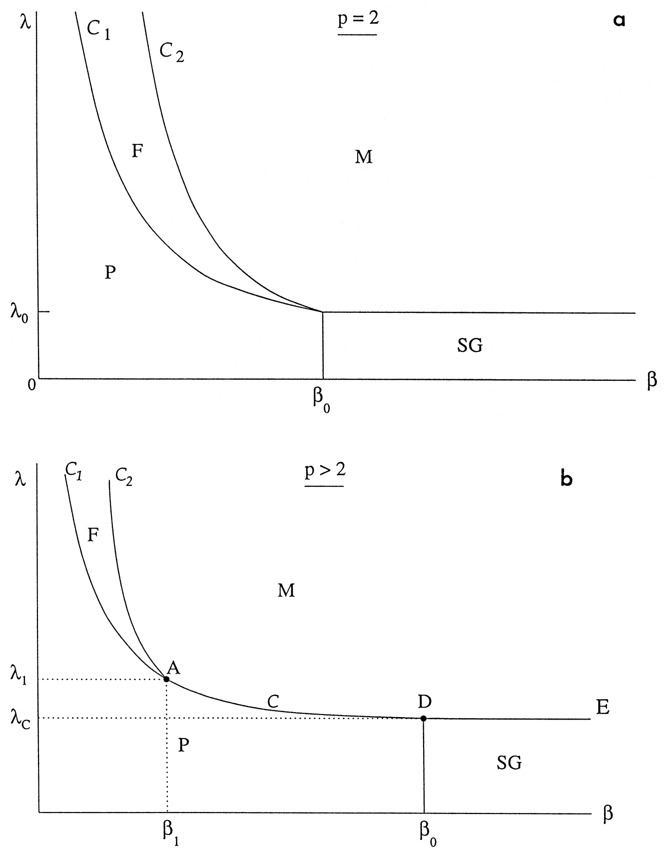

Fig. 2. (a) The phase-diagram for $p=2$; (b) the phase-diagram for $p > 2$.

lines $\mathscr{C}_1$, $\mathscr{C}$ and $\lambda=\lambda_c$ whereas it is continuous across the curve $\mathscr{C}_2 A$. In the paramagnetic region (P), $m=0$ but the free-energy depends on β and in the ferromagnetic region (F) $m(\beta, \lambda) \neq 0$ while in the Spin Glass phase (SG) $m=0$ with the free-energy remaining constant. In the mixed phase (M), $m \neq 0$ and the free-energy depends only on λ.

The computation of the free energy involves large deviation theory. First we write the partition function as an integral with respect to measures defined in terms of the spin-glass on a Cayley tree with an external magnetic field. Then we show that these a priori measures satisfy the large deviation principle (LDP) by first calculating the cumulant generating function. This is done in Section 2, using an extension of the martingale approach of ref. 1. It is well-known that the existence of the cumulant generating function implies the LDP for level-I measures (see refs. 15 and 11). We compute the corresponding rate function as a Legendre transform of this cumulant generating function in Section 3. By Varadhan's theorem we can then write a variational expression for the free energy density. This expression is analyzed in Section 4 for the cases $p=2$ and $p>2$ respectively. Exact expressions for the free energy in the various regions of phase diagram are derived.

2. THE CUMULANT GENERATING FUNCTION

2.1. Definitions

Let the configuration space be the set X^n of all sequences $\{\sigma_i\}_{i=1}^n$ with $\sigma_i \in X=\{-1, 1\}$. Let $\mu(\sigma_i=+1)=\mu(\sigma_i=-1)=1/2$ so that the a-priori probability of each configuration of spin variables is $\mu_n=1/2^n$. Now the partition function (1.4) can be written as

$$\mathscr{Z}_n = \sum_{\{\sigma_j\}_{j=1}^n} \exp\left\{\beta n\left[\frac{1}{n}\sum_{j=1}^n V_{j,(\sigma)^j} + \lambda\left(\frac{1}{n}\sum_{j=1}^n \sigma_j\right)^p\right]\right\} \tag{2.1}$$

Note that since $V_{j,(\sigma)^j}$ depends on all the previous σ_k, $k \leqslant j$, performing the above summation over $\{\sigma_j\}$ is not straight-forward. So, we will exploit martingale properties[17] related to $\mathscr{Z}_n$.

We write the free energy (1.5) as

$$-\beta f(\beta, \lambda, \gamma) = \lim_{n\to\infty} \frac{1}{n} \log \int_{X_n} e^{-\beta\mathscr{H}} \mu_n(d\sigma) + \log 2 \tag{2.2}$$

Also define the observables $V_n: X_n \to \mathbb{R}$, $m_n: X_n \to \mathbb{R}$ by

$$V_n(\sigma) = \frac{1}{n} \sum_{j=1}^{n} V_{j,(\sigma)^j}, \qquad m_n(\sigma) = \frac{1}{n} \sum_{j=1}^{n} \sigma_j \tag{2.3}$$

Notice that the partition function (2.1) only depends on these two variables so that (2.2) can be rewritten as an integral with respect to the distribution N_n of $W_n = (V_n(\sigma), m_n(\sigma))$, i.e., the image measure[12] on $\mathbb{R}^2$ induced by the map W_n:

$$-\beta f(\beta, \lambda, \gamma) = \lim_{n\to\infty} \frac{1}{n} \log \int_{\mathbb{R}^2} \mathrm{e}^{n\beta(v+\lambda m^p)} N_n(dv, dm) + \log 2 \tag{2.4}$$

We wish to compute this limit and show that it converges almost surely with respect to the distribution of the random variables $V_{j,(\sigma)^j}$, which we also denote by ω. We do this by first computing the **cumulant generating function** $C(t_1, t_2)$ defined by

$$C(t_1, t_2) = \lim_{n\to\infty} \frac{1}{n} \log \int e^{t_1 v + t_2 m} N_n(dv, dm) \tag{2.5}$$

This will enable us to compute the corresponding rate function, i.e.,

$$I(v, m) = \sup_{t_1, t_2} \{t_1 v + t_2 m - C(t_1, t_2)\} \tag{2.6}$$

and to apply Varadhan's Theorem[11, 15] to get

$$-\beta f(\beta, \lambda, \gamma) = \sup_{v, m} \{\beta(v + \lambda m^p) - I(v, m)\} + \log 2 \tag{2.7}$$

Denote

$$\tilde{\mathscr{Z}}_n(t_1, t_2) = \sum_{\{\sigma_j\}_{j=1}^n} \mathrm{e}^{t_1 \sum_{j=1}^n V_{j,(\sigma)^j} + t_2 \sum_{j=1}^n \sigma_j} \tag{2.8}$$

and define

$$v^n = \{V_{j,(\sigma)^j};\ 1 \leqslant k \leqslant 2^j,\ 1 \leqslant j \leqslant n\} \tag{2.9}$$

which denotes the set of all the random variables $V_{j,(\sigma)^j}$ between generation 1 and n. Notice that the cumulant generating function (2.5) can be written as

$$C(t_1, t_2) = \lim_{n \to \infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1, t_2) - \log 2 \tag{2.10}$$

Define

$$\Phi(t_1, t_2) = \cosh(t_2)\, \mathbb{E}[e^{t_1 V}] \tag{2.11}$$

and

$$M_n(t_1, t_2) = \frac{\tilde{\mathscr{Z}}_n(t_1, t_2)}{(2\Phi(t_1, t_2))^n} \tag{2.12}$$

where $\mathbb{E}$ denotes the expectation with respect to the random variables V.

2.2. Martingale Results

Proposition 2.1. $\{M_n\}_{n=1}^{\infty}$ is a martingale with respect to the increasing family of random variables $\{v^n\}_{n=1}^{\infty}$, that is,

$$\mathbb{E}(M_{n+1} \mid v^n) = M_n \tag{2.13}$$

Proof. Write

$$V_{n+1,(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})} = \begin{cases} V_1; & \sigma_{n+1} = +1 \\ V_2; & \sigma_{n+1} = -1 \end{cases}$$

$$\tilde{\mathscr{Z}}_{n+1}(t_1, t_2) = \sum_{\{\sigma_j\}_{j=1}^n} \exp\left[t_1 \sum_{j=1}^n V_{j,(\sigma)^j} + t_2 \sum_{j=1}^n \sigma_j \right] \times \sum_{\sigma_{n+1} \in \{-1, 1\}} \exp[t_1 V_{n+1,(\sigma)^{n+1}} + t_2 \sigma_{n+1}] \tag{2.14}$$

Taking the expectation with respect to $V_{n+1,\sigma}$,

$$\begin{aligned} \mathbb{E}[\tilde{\mathscr{Z}}_{n+1}(t_1, t_2) \mid v^n] &= \tilde{\mathscr{Z}}_n(t_1, t_2)\, \mathbb{E}[e^{t_1 V_1 + t_2} + e^{t_1 V_2 - t_2}] \\ &= \tilde{\mathscr{Z}}_n(t_1, t_2)\, 2\cosh(t_2)\, \mathbb{E}(e^{t_1 V}) \end{aligned} \tag{2.15}$$

Dividing by $(2\Phi(t_1, t_2))^{n+1}$ the result follows. ∎

Remark 2.1. 1. $\mathbb{E}[M_n(t_1, t_2)] = 1$.

2. As in ref. 1, if $M_\infty > 0$ with probability 1 then we have

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1, t_2) = \log 2\Phi(t_1, t_2) \qquad \text{a.s.} \tag{2.16}$$

Lemma 2.1. For any fixed t_1, t_2,

$$\mathbb{P}[M_\infty(t_1, t_2) = 0] = 0 \qquad \text{or} \qquad 1 \tag{2.17}$$

Proof. Let L_n (resp. R_n) denote the set of paths of length n which start with a branch in the left (resp. right) direction. Then we have

$$M_n(t_1, t_2) = (2\Phi(t_1, t_2))^{-n} \Bigg[e^{t_1 V_{11} + t_2 \sigma_1} \sum_{(\sigma)^j \in L_n} e^{t_1 \sum_{j=1}^n V_{j,(\sigma)^j} + t_2 \sum_{j=1}^n \sigma_j} + e^{t_1 V_{12} + t_2 \sigma_2} \sum_{(\sigma)^j \in R_n} e^{t_1 \sum_{j=1}^n V_{j,(\sigma)^j} + t_2 \sum_{j=1}^n \sigma_j} \Bigg] \tag{2.18}$$

The event $\{\lim_{n\to\infty} M_n(t_1, t_2) = 0\}$ is independent of V_{11} and V_{12}. Hence it is independent of v^2. Similarly it is independent of v^p for every p. Hence the result follows by Kolmogorov's 0,1-Law. ∎

Remark 2.2. If $\mathbb{P}[M_\infty = 0] = 1$ then $\mathbb{E}[M_\infty] = 0$. Therefore, if we know that $\mathbb{E}[M_\infty(t_1, t_2)] > 0$ then $\mathbb{P}[M_\infty(t_1, t_2) = 0] = 1$ is impossible. As in ref. 1, we prove that

$$\sup_{n \geqslant 1} \mathbb{E}[M_n^\alpha(t_1, t_2)] < \infty \qquad \text{for some} \quad \alpha > 1 \tag{2.19}$$

from which it follows that $\mathbb{E}[M_\infty(t_1, t_2)] = 1$.

The proof of the next lemma takes a similar reasoning as that of ref. 1. We refer to ref. 16 for the proof.

Lemma 2.2.

$$\mathbb{E}[M_{n+1}^2(t_1, t_2) \mid v^n] = M_n^2(t_1, t_2) + \lambda(t_1, t_2)^n \, [\lambda(t_1, t_2) - \lambda(0, t_2)] \, M_n(2t_1, 2t_2) \tag{2.20}$$

where

$$\lambda(t_1, t_2) = \frac{\Phi(2t_1, 2t_2)}{2\Phi^2(t_1, t_2)} \tag{2.21}$$

Remark 2.3. Taking expectations on both sides

$$\mathbb{E}(M_{n+1}^2)=\mathbb{E}(M_n^2)+\lambda(t_1, t_2)^n\,[\lambda(t_1, t_2)-\lambda(0, t_2)] \tag{2.22}$$

and iterating we find

$$\mathbb{E}(M_n^2)=\lambda(t_1, t_2)+\tfrac{1}{2}\operatorname{sech}^2(t_2)+[\lambda(t_1, t_2)-\lambda(0, t_2)]\sum_{k=1}^{n-1}\lambda(t_1, t_2)^k \tag{2.23}$$

Hence we conclude that

$$\sup_{n\geqslant 1}\mathbb{E}[M_n^2(t_1, t_2)] < \infty \qquad \text{whenever} \quad \Phi(2t_1, 2t_2) < 2\Phi^2(t_1, t_2) \tag{2.24}$$

The following lemma was proven in ref. 1:

Lemma 2.3. For any finite set of real numbers $\{x_1, ..., x_n\}$, the function

$$g(\tau)=\frac{1}{\tau}\log\sum_{j=1}^{n}\mathrm{e}^{\tau x_j} \tag{2.25}$$

is decreasing and convex in τ.

Proposition 2.2. Define

$$F(t_1, t_2)=\log[2\Phi(t_1, t_2)] \tag{2.26}$$

If for any given t_1, t_2, there exists $\alpha > 1$ such that

$$F(\alpha t_1, \alpha t_2) < \alpha F(t_1, t_2) \tag{2.27}$$

then

$$\sup_{n\geqslant 1}\mathbb{E}[M_n^\alpha(t_1, t_2)] < \infty \tag{2.28}$$

Proof. Take $1 < \alpha < 2$. By Hölder's inequality

$$\begin{aligned}\mathbb{E}[M_{n+1}^\alpha \mid v^n] &\leqslant (\mathbb{E}[M_{n+1}^2 \mid v^n])^{\alpha/2}\\ &\leqslant M_n^\alpha+[\lambda(t_1, t_2)-\lambda(0, t_2)]^{\alpha/2}\,\lambda(t_1, t_2)^{n\alpha/2}\,M_n^{\alpha/2}(2t_1, 2t_2)\end{aligned} \tag{2.29}$$

By Lemma 2.3 we have $\tilde{\mathscr{Z}}_n^{1/2}(2t_1, 2t_2) \leqslant \tilde{\mathscr{Z}}_n^{1/\alpha}(\alpha t_2, \alpha t_1)$ for $0 < \alpha < 2$. Hence

$$M_n^{1/2}(2t_1, 2t_2) = \frac{\tilde{\mathscr{Z}}_n^{1/2}(2t_1, 2t_2)}{(2\Phi(2t_2 1, 2t_2))^{n/2}}$$
$$\leqslant M_n^{1/\alpha}(\alpha t_1, \alpha t_2)\left[\frac{2\Phi(\alpha t_1, \alpha t_2)}{(2\Phi(2t_1, 2t_2))^{\alpha/2}}\right]^{n/\alpha} \tag{2.30}$$

Inserting this in (2.29) we get

$$\mathbb{E}[M_{n+1}^{\alpha} \mid v^n] \leqslant M_n^{\alpha}(t_1, t_2) + [\lambda(t_1, t_2) - \lambda(0, t_2)]^{\alpha/2}\, \lambda(t_1, t_2)^{n\alpha/2}$$
$$\times M_n(\alpha t_1, \alpha t_2)\left[\frac{2\Phi(\alpha t_1, \alpha t_2)}{(2\Phi(t_1, t_2))^{\alpha}}\right]^n \tag{2.31}$$

Taking expectations, iterating as in (2.23) and substituting for $\lambda(t_1, t_2)$ we find

$$\mathbb{E}[M_{n+1}^{\alpha}] \leqslant \mathbb{E}[M_1^{\alpha}] + [\lambda(t_1, t_2) - \lambda(0, t_2)]^{\alpha/2} \sum_{k=1}^{n} \left[\frac{2\Phi(\alpha t_1, \alpha t_2)}{(2\Phi(t_1, t_2))^{\alpha}}\right]^k \tag{2.32}$$

This proves the proposition. ▮

Remark 2.4. 1. It follows from Hölder's inequality that $F(t_1, t_2)$ is a convex function. So, (2.27) implies that

$$\frac{d}{d\alpha}\frac{1}{\alpha} F(\alpha t_1, \alpha t_2)|_{\alpha=1} = \lim_{\alpha \to 1} \frac{(F(\alpha t_1, \alpha t_2)/\alpha) - F(t_1, t_2)}{\alpha - 1} < 0 \tag{2.33}$$

On the other hand if (2.33) holds, that is if

$$\frac{d}{d\alpha}\frac{1}{\alpha} F(\alpha t_1, \alpha t_2)|_{\alpha=1} = -F(t_1, t_2) + \frac{d}{d\alpha} F(\alpha t_1, \alpha t_2)|_{\alpha=1} < 0 \tag{2.34}$$

then there exists $\alpha > 1$ such that (2.27) holds.

2. In the following we assume that V has a Gaussian distribution with zero mean and variance $1/\gamma$. In that case

$$F(t_1, t_2) = \log[2\cosh(t_2)] + \frac{t_1^2}{2\gamma} \tag{2.35}$$

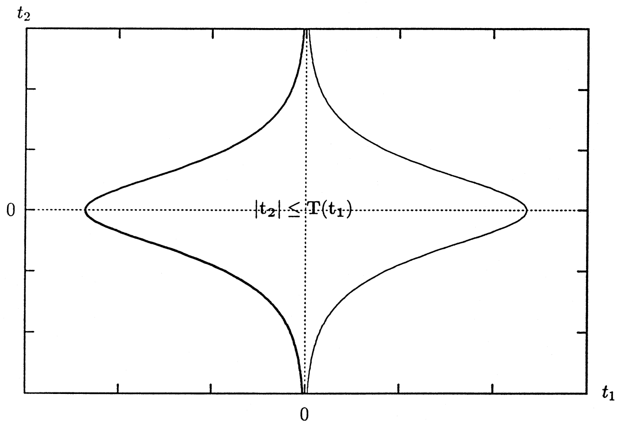

Fig. 3. In the region $|t_1| < B(t_2)$, there is $\alpha > 1$ such that $n \geqslant 1 \mathbb{E}[M_n^\alpha(t_1, t_2)] < \infty$.

and (2.34) reduces to

$$|t_1| < \{2\gamma[\log(2\cosh(t_2)) - t_2 \tanh(t_2)]\}^{1/2} =: B(t_2) \tag{2.36}$$

Also define $\beta_0 := B(0)$. We shall use these definitions throughout the paper hereafter. For future use we write $B^{-1} = T$. It will be convenient to keep the graph of $B(t_2)$ in mind as as proceed with proofs of future results (see Fig. 3).

2.3. Existence of the Cumulant Generating Function

Theorem 2.1. Assume that V takes a Gaussian distribution with zero mean and variance $1/\gamma$ and define

$$\Gamma = \{(t_1, t_2) \in \mathbb{R}^2 : |t_2| < T(t_1)\}$$

Then the following limit holds almost surely:

$$\lim_{n \to \infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1, t_2) = \begin{cases} F(t_1, t_2); & (t_1, t_2) \in \Gamma \\ t_1 F(\bar{t}_1, \bar{t}_2)/\bar{t}_1; & (t_1, t_2) \in \Gamma^{\mathrm{C}} \end{cases} \tag{2.37}$$

where $\bar{t}_1, \bar{t}_2$ are the solutions of the equations

$$\frac{\bar{t}_2}{\bar{t}_1} = \frac{t_2}{t_1} \qquad \text{and} \qquad \bar{t}_1 = B(\bar{t}_2) \tag{2.38}$$

Proof. (i) $(t_1, t_2) \in \Gamma$: Now proposition (2.2) applies and we have $\mathbb{E}[M_\infty] = 1$ (see Remark 2.2). Hence using lemma (2.1), the result follows by (2.16). We restate this: Define

$$\Omega_{(t_1, t_2)} = \left\{ \omega \in \Omega : \lim_{n \to \infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(t_1, t_2) = F(t_1, t_2) \right\} \tag{2.39}$$

Then

$$\mathbb{P}[\Omega_{(t_1, t_2)}] = 1 \qquad \text{for each} \quad (t_1, t_2) \in \Gamma \tag{2.40}$$

But we need a stronger result, namely

$$\mathbb{P}\left[\bigcap_{(t_1, t_2) \in \Gamma} \Omega_{(t_1, t_2)} \right] = 1 \tag{2.41}$$

that is, the exceptional nullset is uniform in (t_1, t_2). The proof of this is given in ref. 16.

(ii) $(t_1, t_2) \in \Gamma^C$: Take any $(t_1, t_2) \in \Gamma^C$ (cf. Fig. 4). Let $(\bar{t}_1, \bar{t}_2)$ be the solution of (2.38) which is obviously unique. Put $\tau = t_1/\bar{t}_1 = t_2/\bar{t}_2 \geqslant 1$. By Lemma 2.3, $\log \tilde{\mathscr{Z}}_n(\tau t_1, \tau t_2)/\tau$ is decreasing and convex in τ. By the decrease

$$\limsup_{n \to \infty} \frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau \bar{t}_1, \tau \bar{t}_2) \leqslant \limsup_{n \to \infty} \frac{1}{n(1-\varepsilon)} \log \tilde{\mathscr{Z}}_n((1-\varepsilon)\,\bar{t}_1, (1-\varepsilon)\,\bar{t}_2)) \tag{2.42}$$

Since $(1-\varepsilon)(\bar{t}_1, \bar{t}_2) \in \Gamma$ we get by letting $\varepsilon \to 0$

$$\limsup_{n \to \infty} \frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau \bar{t}_1, \tau \bar{t}_2) \leqslant F(\bar{t}_1, \bar{t}_2) \tag{2.43}$$

On the other hand, by the convexity we find

$$\begin{aligned} \liminf_{n \to \infty} \frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau \bar{t}_1, \tau \bar{t}_2) \geqslant {} & \liminf_{n \to \infty} \frac{1}{n(1-\varepsilon)} \log \tilde{\mathscr{Z}}_n((1-\varepsilon)\,\bar{t}_1, (1-\varepsilon)\,\bar{t}_2) \\ & + \liminf_{n \to \infty} \frac{d}{d\tau} \left[\frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau \bar{t}_1, \tau \bar{t}_2) \right]_{\tau = 1-\varepsilon} (\tau - 1 + \varepsilon) \end{aligned}$$

Since the sequence of convex functions

$$\frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau t_1, \tau t_2)$$

converges to $F(\tau t_1, \tau t_2)/\tau$ a.s. for $\tau(t_1, t_2) \in \Gamma$ (by the proof in part (i) of the theorem), their derivatives converge to the limit

$$\frac{d}{d\tau}\left[\frac{1}{\tau} F(\tau t_1, \tau t_2)\right]$$

Hence

$$\liminf_{n\to\infty} \frac{1}{n}\frac{d}{d\tau}\left[\frac{1}{\tau}\log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2)\right]_{\tau=1-\varepsilon} = \frac{d}{d\tau}\left[\frac{1}{\tau} F(\tau\bar{t}_1, \tau\bar{t}_2)\right]_{\tau=1-\varepsilon} \tag{2.44}$$

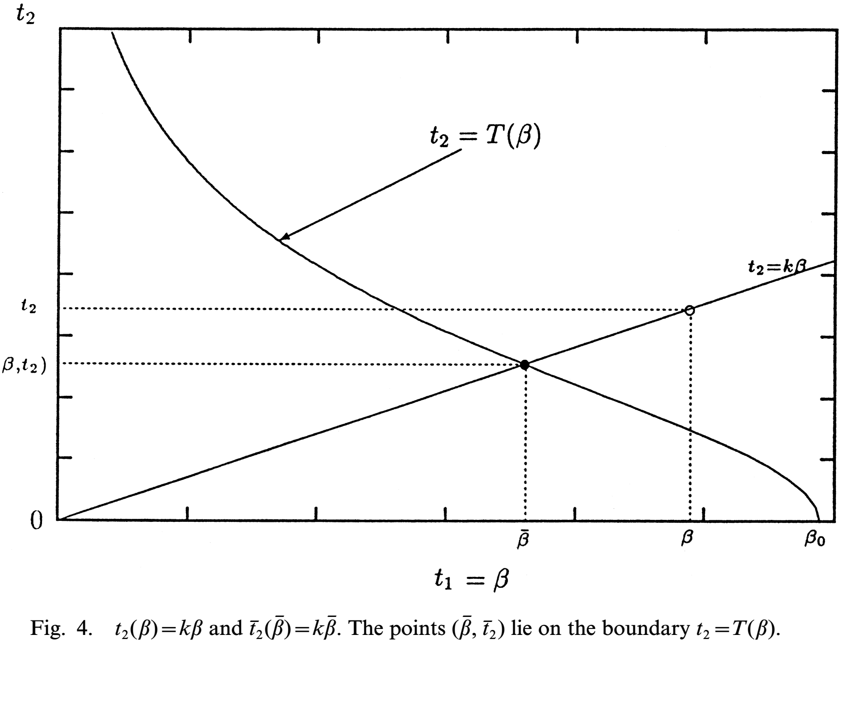

Fig. 4. $t_2(\beta) = k\beta$ and $\bar{t}_2(\bar{\beta}) = k\bar{\beta}$. The points $(\bar{\beta}, \bar{t}_2)$ lie on the boundary $t_2 = T(\beta)$.

Moreover since (2.34) is equivalent to (2.36),

$$\frac{d}{d\tau}\left[\frac{1}{\tau}F(\tau t_1, \tau t_2)\right]_{\tau=1} = 0 \qquad \text{at} \quad (\bar{t}_1, \bar{t}_2) \tag{2.45}$$

It follows that

$$\liminf_{n\to\infty} \frac{1}{n\tau} \log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2) \geqslant F(\bar{t}_1, \bar{t}_2) \qquad \text{a.s.} \tag{2.46}$$

as $\varepsilon \to 0$. Hence we have

$$\lim_{n\to\infty} \frac{1}{n} \log \tilde{\mathscr{Z}}_n(\tau\bar{t}_1, \tau\bar{t}_2) = \tau F(\bar{t}_1, \bar{t}_2) \qquad \text{a.s.} \quad \blacksquare \tag{2.47}$$

Corollary 2.1.1. There exists a uniform null set $\mathcal{N}$ such that the cumulant generating function

$$C(t_1, t_2)(\omega) = \begin{cases} F(t_1, t_2) - \log 2; & (t_1, t_2) \in \Gamma \\ t_1 F(\bar{t}_1, \bar{t}_2)/\bar{t}_1 - \log 2; & (t_1, t_2) \in \Gamma^{\mathrm{C}} \end{cases} \tag{2.48}$$

is defined and exists for all (t_1, t_2) if $\omega \notin \mathcal{N}$.

N.B: In the second case $\bar{t}_1$ and $\bar{t}_2$ has to be determined so that (2.38) is satisfied.

3. VARIATIONAL FORMULA

3.1. The Rate Function

Lemma 3.1. Let

$$I^\beta(m) := \sup_t \{tm - C(\beta, t)\} \tag{3.1}$$

Then the free-energy expression (2.7) can be written as

$$-\beta f(\beta, \lambda) = \log 2 + \sup_m \{\lambda\beta m^p - I^\beta(m)\} \tag{3.2}$$

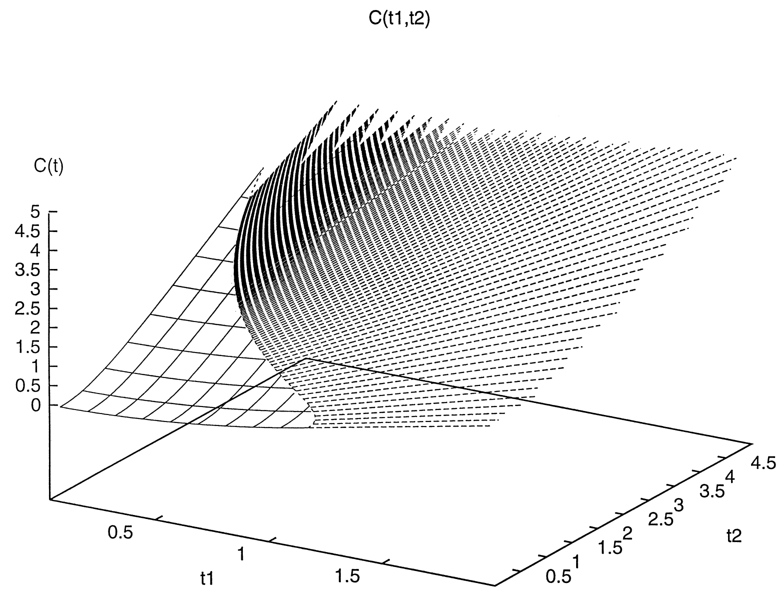

Fig. 5. $C(t_1, t_2)$ at $\gamma=1$ for given $k=t_2/t_1$. Notice it's linear behaviour beyond the region Γ along the radical lines.

Proof. Notice that the free-energy expression (2.7) can be written as

$$-\log 2-\beta f(\beta, \lambda)=\sup_m \{\beta\lambda m^p+\sup_v [\beta v-I(v, m)]\} \tag{3.3}$$

where the second supremum is the Legendre transform of I with respect to the first variable. We can also write the rate function (2.6) as

$$I(v, m)=\sup_{t_1} \{t_1 v-(-I^{t_1}(m))\} \tag{3.4}$$

By convexity of $-I^{t_1}(m)$ with respect to t_1 (since C is convex) we can invert (3.1) to get

$$-I^{t_1}(m)=\sup_v \{t_1 v-I(v, m)\} \tag{3.5}$$

Inserting this in (3.3) yields the lemma. ∎

Remark 3.1. 1. Notice that F is symmetric in both variables (provided the distribution of V is symmetric). In particular

$$C(-t_1, t_2)=C(t_1, t_2)=C(t_1, -t_2)$$

2. Since t_1 should take the same sign as v in the following, we have,

$$I(v, m) = \sup_{t_1 \in \mathbb{R}} \{t_1 v + I^{t_1}(m)\} = \sup_{t_1 > 0} \{t_1 \ |v| + I^{t_1}(m)\} = \sup_{t_1 \in \mathbb{R}} \{t_1 \ |v| + I^{t_1}(m)\} \tag{3.6}$$

and hence $I(v, m) = I(|v|, m)$. Also, it follows by a similar reasoning considering

$$-I^{t_1}(m) = \sup_{t_2} \ \{t_2 m - C(t_1, t_2)\} \tag{3.7}$$

that $I(v, m) = I(v, -m)$ and hence we have

$$I(v, m) = I(|v|, |m|) \tag{3.8}$$

Therefore it suffices to consider only $t_1 = \beta > 0$ and $m > 0$, in all the derivations that follow. We find $t_1 = \beta$ and therefore it is convenient to write t instead of t_2 hereafter.

Proposition 3.1. Let $m(\beta) = \tanh[T(\beta)]$, $\bar{\beta}(m) = \{2\gamma \log 2 - I_0(m)\}^{1/2}$ and $I_0(m) = [(1+m) \log(1+m) + (1-m) \log(1-m)]/2$. Then

$$I^{\beta}(m) = \begin{cases} I_1^{\beta}(m) := I_0(m) - \dfrac{\beta^2}{2\gamma}; & 0 \leqslant |m| \leqslant m(\beta) \\ I_2^{\beta}(m) := -\dfrac{\beta\bar{\beta}}{\gamma} + \log 2; & m(\beta) < |m| < 1 \\ \infty; & \text{otherwise} \end{cases} \tag{3.9}$$

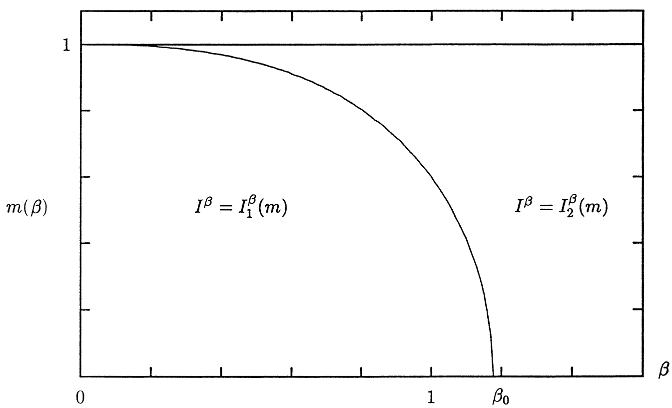

Fig. 6. The curve and the line show the boundary of the domain of validity of $I^{\beta}(m)$.

Proof. Notice that since $I^{\beta}(m)=\max\{I_1^{\beta}, I_2^{\beta}\}$ one has to determine which one of I_1^{β} and I_2^{β} dominates and when that happens. This is done by solving the one-dimensional variational problem where one simply differentiates the two forms of $C(\beta, t)$ in the regions Γ and Γ^C (cf. (2.48)) with respect to t. ∎

Remark 3.2. The boundary of the domain of validity of $I_1^{\beta}(m)$ is shown in the Fig. 6 which is obtained by solving the explicit equation $B(t)=\beta$ for t and then taking $m(\beta)=\tanh(t)$.

4. FREE ENERGY AND THE PHASE DIAGRAM

We now discuss the phase diagram as depicted in Fig. 2(a) i.e., for the case $p=2$. We define the graphs $\lambda=\mathscr{C}_1(\beta)$, $\lambda=\mathscr{C}_2(\beta)$ by

$$\begin{aligned}\mathscr{C}_1(\beta)&=1/(2\beta)\\ \mathscr{C}_2(\beta)&=\frac{T(\beta)}{2\beta m(\beta)}\end{aligned} \tag{4.1}$$

where $m(\beta)=\tanh[T(\beta)]$.

The proof of the following lemma is given in ref. 16:

Lemma 4.1. If $\lambda > \lambda_0$ and $\lambda > \mathscr{C}_2(\beta)$ the equation $m=\tanh[2\bar{\beta}(m)\,\lambda m]$ has exactly one positive solution on $[0, 1]$ which increases as λ increases, and if $\lambda \leqslant \lambda_0$ the only solution is $m=0$.

Theorem 4.1. Let $p=2$. Then the free energy is given by

$$f(\beta,\lambda)=\begin{cases} f_P(\beta)\equiv -\dfrac{\beta}{2\gamma}-\dfrac{1}{\beta}\log 2; \\ \qquad \lambda\leqslant \mathscr{C}_1(\beta),\ 0<\beta<\beta_0 \\ f_F(\beta,\lambda)\equiv -\lambda\bar{m}(\beta,\lambda)^2+\dfrac{1}{\beta}I_0(\bar{m}(\beta,\lambda))-f_P(\beta); \\ \qquad \mathscr{C}_1(\beta)\leqslant\lambda\leqslant\mathscr{C}_2(\beta),\ 0<\beta<\beta_0 \\ f_M(\lambda)\equiv -\lambda m_\lambda^2-\dfrac{\bar{\beta}(m_\lambda)}{\gamma}; \\ \qquad \lambda\geqslant\mathscr{C}_2(\beta),\ \beta\leqslant\beta_0 \quad \text{or} \quad \lambda\geqslant\lambda_0,\ \beta>0 \\ f_{SG}\equiv -\dfrac{\beta_0}{\gamma}; \qquad \lambda\leqslant\lambda_0,\ \beta\geqslant\beta_0 \end{cases} \tag{4.2}$$

and the corresponding phase diagram is given by Fig. 2(a). Here

$$\bar{m}(\beta, \lambda) = \tanh[2\beta\lambda\bar{m}(\beta, \lambda)] \tag{4.3}$$

$$m_\lambda = \tanh[2\bar{\beta}(m_\lambda)\, \lambda m_\lambda] \tag{4.4}$$

$\lambda_0 = \mathscr{C}_1(\beta_0) = 1/(2\beta_0)$ and $\bar{\beta}(m) = \sqrt{2\gamma[\log 2 - I_0(m)]}$.

Proof. Define $g_i(\beta, \lambda; m) = \beta\lambda m^2 - I_i^\beta(m)$, $i = 1, 2$, where I_i^β are the two forms of $I^\beta(m)$ as defined in (3.9). Clearly,

$$\partial g_1/\partial m = 0 \Leftrightarrow m = \tanh(2\beta\lambda m) \tag{4.5}$$

and, using

$$\bar{\beta}'(m) = -\frac{\gamma \tanh^{-1}(m)}{\bar{\beta}(m)} \tag{4.6}$$

$$\partial g_2/\partial m = 0 \Leftrightarrow m = \tanh(2\bar{\beta}(m)\, \lambda m) \tag{4.7}$$

These are just (4.3) and (4.4) and it remains to determine which case applies in various regions of the β, λ-plane.

First suppose that $0 < \beta < \beta_0$. If $\lambda \leqslant C_1(\beta)$ then (4.3) has only the zero solution and the maximizer is attained at $m = 0$. If $\mathscr{C}_1(\beta) \leqslant \lambda \leqslant \mathscr{C}_2(\beta)$ then the maximum is attained at the positive solution $m = \bar{m}(\beta, \lambda)$ of (4.5), that is (4.3) holds and $I^\beta(m) = I_1^\beta(m)$. Indeed, $\bar{m}(\beta, \lambda) \leqslant m(\beta)$ since $\lambda \leqslant \mathscr{C}_2(\beta)$ and $\bar{m}$ increases with λ.

If $\lambda \geqslant \mathscr{C}_2(\beta)$ then $g_1(\beta, \lambda; m)$ is increasing in m for $m \leqslant m(\beta)$ so that its maximum is attained at $m = m(\beta)$. At that point $g_1 = g_2$ so that the maximum is always attained for $m \geqslant m(\beta)$ and $f(\beta, \lambda) = g_2(\beta, \lambda; m)$.

By Lemma 4.1, Eq. (4.7) has a positive solution $m_\lambda \geqslant m(\beta)$ which corresponds to the maximum. The free energy follows by insertion: $f(\beta, \lambda) = -\lambda m_\lambda^2 + I_2^\beta(m_\lambda) - \log 2 = f_M(\lambda)$.

Next consider the case $\beta > \beta_0$. Then $I_1^\beta(m)$ does not apply so that $f(\beta, \lambda) = g_2(\beta, \lambda; m)$, where m is the maximizer. Clearly, if $\lambda \leqslant \lambda_0$ then $m = 0$ by Lemma 4.1 and $f(\beta, \lambda) = \beta^{-1}(I_2^\beta(0) - \log 2) = f_{SG}$ whereas if $\lambda \geqslant \lambda_0$ the maximizer is given by the unique positive solution of (4.4) and $f(\beta, \lambda)$ is again given by $f_M(\lambda)$. ∎

The phase diagram for the case $p > 2$ can be determined in the same lines as for the case $p = 2$, the details are given in ref. 16. We state the main theorem:

Theorem 4.2. Put $g_i(\beta, \lambda; m) = \beta\lambda m^p - I_i^\beta(m)$; $i = 1, 2$. Let $\bar{m}_1(\beta, \lambda)$ be the unique positive solution of $\partial g_1/\partial m = 0$ such that $g_1(\beta, \lambda; m) \geqslant g_1(\beta, \lambda; 0)$ where $0 \leqslant \bar{m}_1(\beta, \lambda) \leqslant m(\beta)$; $\bar{m}_2(\lambda)$ be that of $\partial g_2/\partial m = 0$ such that $g_2(\beta, \lambda; m) \geqslant g_2(\beta, \lambda; 0)$ where $\bar{m}_2(\lambda) \geqslant m(\beta)$ for $0 \leqslant \beta \leqslant m(\beta)$ and $\bar{m}(\lambda)$ be that of $\partial g_2/\partial m = 0$ such that $g_2(\beta, \lambda; m) \geqslant g_1(\beta, \lambda; 0)$ where $\bar{m}(\lambda) \geqslant m(\beta_1)$ for $\beta > \beta_1 = \bar{\beta}(\bar{m}(\beta, \lambda))$.

Let m_p be the critical value of $\bar{m}_1(\beta, \lambda)$ given by $g_1(\beta, \lambda; m) = g_1(\beta, \lambda; 0)$. Moreover, let the curves $\mathscr{C}_1(\beta)$ and $\mathscr{C}_2(\beta)$ be defined by

$$\mathscr{C}_1(\beta) = \frac{\tanh^{-1}(m_p)}{p\beta m_p^{p-1}} \qquad \text{and} \qquad \mathscr{C}_2(\beta) = \frac{\tanh^{-1}(m(\beta))}{p\beta (m(\beta))^{p-1}} \tag{4.8}$$

respectively. Then, for $0 < \beta \leqslant \beta_1$,

$$-\beta f(\beta, \lambda) - \log 2 = \begin{cases} -I_1^\beta(0) = \dfrac{\beta^2}{2\gamma} & \text{if } \lambda \leqslant \mathscr{C}_1(\beta) \\ \beta\lambda\bar{m}_1(\beta, \lambda) - I_1^\beta(\bar{m}_1(\beta, \lambda)) & \text{if } \mathscr{C}_1(\beta) \leqslant \lambda \leqslant \mathscr{C}_2(\beta) \\ \beta\lambda\bar{m}_2(\lambda) - I_2^\beta(\bar{m}_2(\lambda)) & \text{if } \lambda > \mathscr{C}_2(\beta) \end{cases} \tag{4.9}$$

The magnetization jumps from 0 to m_p across $\mathscr{C}_1(\beta)$ but is continuous across $\mathscr{C}_2(\beta)$.

For $\beta \geqslant \beta_0$,

$$-\beta f(\beta, \lambda) - \log 2 = \begin{cases} -I_2^\beta(0) = \dfrac{\beta\beta_0}{\gamma} - \log 2 & \text{if } \lambda \leqslant \lambda_c \\ \beta\lambda\bar{m}(\lambda)^p - I_2^\beta(\bar{m}(\lambda)) & \text{if } \lambda \geqslant \lambda_c \end{cases} \tag{4.10}$$

The magnetization jumps from 0 to m_c at $\lambda = \lambda_c$.

For $\beta_1 \leqslant \beta \leqslant \beta_0$ there is a curve $\mathscr{C}(\beta)$ given by

$$\lambda = \mathscr{C}(\beta) \Leftrightarrow -I_1^\beta(0) = \beta\lambda\bar{m}_2(\lambda)^p - I_2^\beta(\bar{m}_2(\lambda)) \tag{4.11}$$

such that $\mathscr{C}(\beta_1) = \mathscr{C}_2(\beta_1) = \lambda_1$, $\mathscr{C}(\beta_0) = \lambda_c$, $\mathscr{C}'(\beta_1) = \mathscr{C}'_1(\beta_1)$ and $\mathscr{C}'(\beta_0) = 0$. The free energy in this case is given by

$$-\beta f(\beta, \lambda) - \log 2 = \begin{cases} -I_1^\beta(0) = \dfrac{\beta^2}{2\gamma} & \text{if } \lambda \leqslant \mathscr{C}(\beta) \\ \beta\lambda\bar{m}(\lambda)^p - I_2^\beta(\bar{m}(\lambda)) & \text{if } \lambda \geqslant \mathscr{C}(\beta) \end{cases} \tag{4.12}$$

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